Game Theory

Introduction

Game Theory is an important field of mathematics which concerns the analysis of game strategies and is applicable to a wide range of disciplines, including economics and politics. Essentially, Game Theory aims to identify complete solutions to games or situations which can outline how an individual can place themselves in the best possible (winning) position. However, this can often prove challenging depending on the simplicity of the game or situation. For example, if we compare the games Connect 4 and Monopoly, these would involve very different solutions.

There are two distinct classifications of Game Theory: Combinatorial Game Theory and Classical Game Theory. Combinatorial Game Theory focuses on the study of two–player games whereby both players know all of the rules and take alternate moves. Furthermore, there are no 'chance' elements to such games and each player can see the moves that have been previously made. In contrast, Classical Game Theory involves players moving or strategizing simultaneously and is characterised by concealed information and elements of chance.

Aim of the Workshop

The aim of this workshop is to introduce students to the concept of Game Theory and the application of **backwards induction** to a combinatorial game to solve various problems. Students use their knowledge of probability and statistics in an attempt to find a solution to a number of games including 'NIM' and 'Guess the Average' game.

Learning objectives

By the end of this workshop students will be able to:

- Discuss the relevance of Game Theory to applications in real life
- Provide a description (in their own words) of backwards induction
- Assign winning and losing positions in the NIM game
- Describe what is meant by 'optimal strategy'

Materials and resources

21 sticks/counters for each pair of students, sheets of paper.

Keywords

Optimal Strategy
A sequence of moves which leads to the best outcome of the game or situation.

Backward induction
A method used to solve sequential games whereby a player works backwards in order to determine the optimal strategy of the game.

Dominant Strategy
A strategy is considered dominant if it earns a player a better payoff in the game, regardless of the other players' actions.

Nash Equilibrium
A set of techniques in which no player can benefit from a change in their strategy as long as all other players' strategies remain unchanged. It is also relevant to economics and was first developed by John Forbes Nash Jr, an American mathematician whose extraordinary life inspired the storyline to the film "A Beautiful Mind".

Game Theory: Workshop Outline

SUGGESTED TIME (TOTAL MINS)	ACTIVITY	DESCRIPTION
5 mins (00:05)	**Introduction to Game Theory**	– Discuss 'Game Theory' as relevant to real life
10 mins (00:15)	**Combinatorial Game Definition and Traits**	– Discuss the features of combinatorial games (see **Appendix – Note 1**) – Task: ask students if a selection of games are/ are not combinatorial and why (see **Appendix – Note 2** for list) – Task: ask students to list and justify extra examples of combinatorial games
10 mins (00:25)	**Introduction to NIM Game**	– Explain the rules of the NIM game (see **Appendix – Note 3**)
10 mins (00:35)	**Optimal strategy**	– Define what is meant by 'optimal strategy' – Ask students if they can come up with an optimal strategy to win the NIM game – Task: Divide students into pairs and ask them to play the game at least 3 times (can take 1, 2 or 3 sticks) – Ask if anyone has come up with a winning theory – Class discussion of ideas
10 mins (00:45)	**Backwards induction**	– Task: Introduce 'backwards induction' whereby students attempt the same game with 4 sticks – Class discussion on solution – Task: Using backwards induction, students play with 21 sticks again and try to identify a pattern or strategy
15 mins (01:00)	**Change the NIM values: can take 1, 3, or 4 sticks at each turn**	– Students play the new game and attempt to come up with an optimal strategy using the same idea of backwards induction as before

SUGGESTED TIME (TOTAL MINS)	ACTIVITY	DESCRIPTION
5 –10 mins (01:10)	**Introduction to the Pirates Puzzle**	– Introduce the Pirate Puzzle (see **Activity Sheet 1**) – Ask five volunteers to come up and demonstrate possible scenarios for the five pirates puzzle – Suggest approach and layout to the problem (see Appendix – **Note 5**)
10–15 mins (01:25)	**Doing the Puzzle**	– Task: **Activity Sheet 2** In pairs, students work together using backwards induction (starting with 2 pirates and working up to 5 pirates) to determine the optimal suggestion for the eldest pirate – Whole class discussion on strategies and solution (as noted in Appendix – **Note 4**)
5 mins (01:30)	**John Nash**	– Briefly discuss John Nash before introducing the Nash equilibrium – (option to play 'A Beautiful Mind' clip (link included in Additional Resources))
10 – 15 mins (1:45)	**Guess the average game**	– Choose 10 volunteers for this activity and explain how the game works (see Appendix – **Note 5**)
5 mins (01:50)	**Discussion and link to Nash equilibrium**	– Explain how the numbers decrease after each iteration due to everyone being aware of the game and decreasing their guess accordingly. The Nash equilibrium is zero
5 mins (01:55)	**Conclusion of Workshop**	– Encourage students to write down three things they have learned from the workshop and encourage students to independently follow up on any questions or queries they might have

Game Theory – Workshop Appendix

Note 1: Criteria for Combinatorial Games

1. Two players who take alternate moves.
2. The rules of the game specify the legal moves for both players to and from each position.
3. There are no elements of chance (e.g. using a die)
4. The game eventually comes to an end.
5. There are no draws and the winner is determined by the player who makes the final move.

Note 2: Examples of Combinatorial games

Combinatorial games: Chess, Checkers, Connect 4, NIM, Go

Non-combinatorial games: Monopoly (has an element of chance), Rock, Paper, Scissors (players take turns simultaneously), soccer (has no legal moves), Xs and Os (can result in a draw)

Note 3: Rules for NIM Game

1. Each pair is provided with 21 sticks which are placed into a row on the table.
2. Students take alternate turns removing sticks from the table
3. Students take a defined number of sticks during their turn (e.g. 1, 2 or 3).
4. The student who picks up the last stick wins the game

(In this game there is a strictly dominant strategy. By this we mean that once the strategy is known to one of the players, there is no way to beat this player. Hence, this optimal strategy strictly dominates any strategy used by the opposing player.)

Solution – (picking up 1, 2, or 3 sticks)
Students can derive the solution by using the blank Activity Sheet 1 and highlighting winning versus losing positions or turns. In this case, all multiples of 4 (4n) are losing positions. Since 21 is not a multiple of 4, the winning strategy is to be the first player to move and to pick up only one stick.

Solution – (picking up 1, 3, or 4 sticks)
Students can derive the solution by using the blank Activity Sheet 1 and highlighting winning versus losing positions or turns. This time the losing positions are multiples of 7 (7n) or multiples of 7 plus 2 (7n + 2). This time, since 21 is a multiple of 7, the winning strategy is to be the second player to move.

Note 4: Solving the Pirate Puzzle

1. Label the pirates A, B, C, D and E for convenience, with A being the eldest.
2. Keep in mind that the eldest pirate wants to pay the other pirates the least amount necessary.
3. Remember the most important thing to each pirate is survival and then coins.
4. Work backwards (as in the NIM game) to decide the best outcome so that the eldest pirate stays alive and gets as many coins as many coins as possible.

	Pirate				
	A	B	C	D	E
1	-	-	-	-	100
2	-	-	-	100	0
3	-	-	99	0	1
4	-	99	0	1	0
5	98	0	1	0	1

Solution:
(You may wish to not provide this suggested table to students during the workshop)

One Pirate: When there is only one pirate (in this case Pirate E), he will get all 100 coins.

Two Pirates: In the case of two pirates D and E, Pirate D will propose to split the coins 100 : 0. His vote (50%) is enough to secure this deal and hence he is not thrown overboard.

Three Pirates: Pirate C (being the eldest of C, D and E) will suggest to split the coins 99 : 0 : 1. Pirate D will vote against this. However, pirate E will accept the offer of getting just 1 coin as he is aware that if he rejects the deal there will be only two pirates left and he will therefore get nothing.

Four Pirates: In the case of four pirates (B, C, D and E), Pirate B will choose to divide the coins 99 : 0 : 1 : 0. By the same reasoning as before, Pirate D will support this offer as he knows that if he rejects it there will be three pirates left and he will then get nothing. Pirate C will always vote against the offer because he knows that if his vote wins, Pirate B will be thrown overboard and he will therefore get 99 coins (as previous in the example). Pirate B will therefore not waste a coin on Pirate C. Likewise Pirate E will also vote against the proposal as he knows that if Pirate B is thrown overboard, he will get 1 coin in the following round (three pirates). However B and D still have 50% of the vote.

Five Pirates: Pirate A will split the coins 98 : 0 : 1 : 0 : 1. By offering 1 coin to Pirate C and E (who would otherwise get nothing) he secures the deal.

Note 5: Guess the Average Game

1. 10 volunteers are chosen. Each volunteer writes a number between 0–100 on a piece of paper but keeps it covered.
2. Volunteers asked to take a guess at what they think half the average of all ten guesses will be.
3. Students think and justify their reasoning to the class. (For example, all answers above 50 can be automatically ruled out. As the highest possible average is 100, and half of that is 50. Now, if everyone playing the game thinks this way, the highest possible average of the guesses is 50. So, half this value would be 25. This means every value above 25 can also be ruled out etc).
4. Calculate the average of the 10 guesses

For example:

If the 10 guesses are: 35, 50, 18, 65, 42, 31, 24, 12, 49, 14

The average is therefore: 35 + 50 + 18 + 65 + 42 + 31 + 24 + 12 + 49 + 14 = 34

So, half the average is 17 and the nearest guess to 17 will win the game.

Sources and Additional Resources

http://mathworld.wolfram.com/GameTheory.html

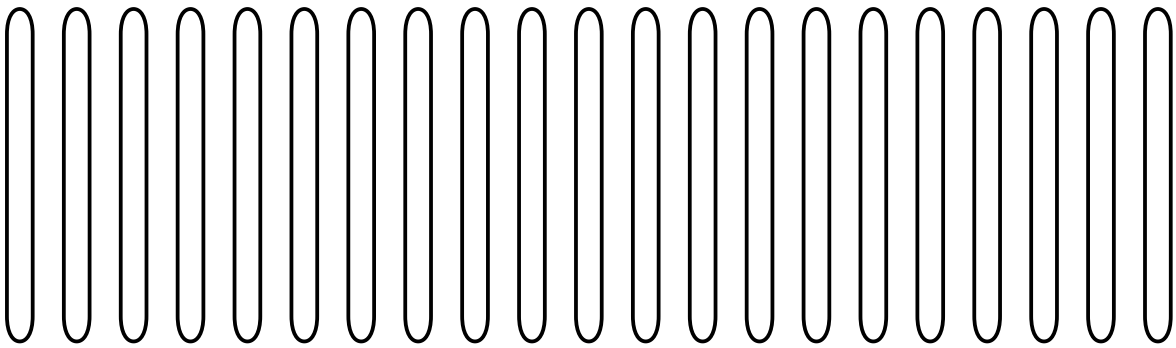
http://www.mathsisfun.com/puzzles/5-pirates.html

https://www.youtube.com/watch?v=LJS7Igvk6ZM (Beautiful Mind Clip)

Freakonomics Podcast: How to Win Games and Beat People

'How to Win Games and Beat People' – Book by Tom Wipple

Game Theory: Activity Sheet 1



Game Theory: Activity Sheet 2

Pirate Puzzle

5 pirates of different ages have a treasure of 100 gold coins.

On their ship, they decide to split the coins using this scheme:

- The oldest pirate always proposes how to share the coins, and ALL pirates (including the oldest) vote for or against it.
- If **50% or more** of the pirates vote for it, then the coins will be shared that way. **Otherwise**, the pirate proposing the scheme will be **thrown overboard**, and the process is repeated with the remaining pirates.
- If a pirate would get the same number of coins if he voted for or against a proposal, he will vote against so that the pirate who proposed the plan will be thrown overboard
- Assuming that all 5 pirates are intelligent, rational, greedy, and do not wish to die, (and are rather good at maths for pirates) what will happen?

Let **A** be the eldest pirate and **E** the youngest. Starting with just 1 pirate and working up to 5, can you fill out the table below to see how many coins the pirates will get? The first one has been completed for you.

	Pirate				
	A	B	C	D	E
1	-	-	-	-	100
2					
3					
4					
5					